# Quark zero modes in intersecting center vortex gauge fields*

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**Abstract**

The zero modes of the Dirac operator in the background of center vortex gauge field configurations in $\mathbb{R}^2$ and $\mathbb{R}^4$ are examined. If the net flux in $D = 2$ is larger than 1 we obtain normalizable zero modes which are mainly localized at the vortices. In $D = 4$ quasi-normalizable zero modes exist for intersecting flat vortex sheets with the Pontryagin index equal to 2. These zero modes are mainly localized at the vortex intersection points, which carry a topological charge of $\pm\frac{1}{2}$. To circumvent the problem of normalizability the space-time manifold is chosen to be the (compact) torus $\mathbb{T}^2$ and $\mathbb{T}^4$, respectively. According to the index theorem there are normalizable zero modes on $\mathbb{T}^2$ if the net flux is non-zero. These zero modes are localized at the vortices. On $\mathbb{T}^4$ zero modes exist for a non-vanishing Pontryagin index. As in $\mathbb{R}^4$ these zero modes are localized at the vortex intersection points.

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# 1 Introduction

There are two fundamental, principally non-perturbative phenomena of the strong interaction which should be explained by QCD: confinement and chiral symmetry breaking. Soon after the advent of QCD chiral symmetry breaking has been explained by assuming that the QCD vacuum contains an ensemble of instantons and anti-instantons [1]. Lattice simulations have, however, shown that instantons cannot be responsible for confinement since they account for only about ten percent of the string tension [2, 3, 4]. On the other hand, recent lattice calculations have produced mounting evidence that confinement is due to the condensation of center vortices in the Yang-Mills vacuum [5, 6]. Since on the lattice the confinement phase transition is observed to occur at the same temperature [7] at which chiral symmetry is restored, one expects that chiral symmetry breaking and confinement are triggered by the same mechanism.

Since instantons cannot explain confinement one wonders whether center vortices are also capable of producing chiral symmetry breaking in addition to confinement. Indeed, it has been shown that in a Yang-Mills ensemble devoid of center vortices the relevant order parameter, the quark condensate vanishes [6]. This does, however, not yet mean that center vortices produce chiral symmetry breaking in the QCD vacuum, since the field-configurations which produce it could be tied or attached to center vortices and simultaneously removed with the latter. It is therefore still an open question whether center vortices produce chiral symmetry breaking and what the underlying mechanism behind this is.

In order to get an idea how chiral symmetry breaking could be produced by center vortices let us first recall how it arises in the instanton picture of the QCD vacuum. In this picture the zero modes of quarks in the instanton background play a crucial role. These zero modes occur due to the topological charge of the instantons and are localized near their topological charge density [8, 9, 10].

In an instanton-anti-instanton ensemble the localized zero modes of the individual instantons spread out in space and form a continuum of quasi zero modes which by the Banks-Casher relation give rise to a non-zero quark condensate /citeZahed. In fact, independent of the instanton picture lattice calculations show a strong correlation between topological charge density and the quark condensate [12, 14]. This correlation is mainly due to the zero modes which exist in topologically non-trivial gauge fields with non-vanishing Pontryagin index due to the Atiyah-Singer index theorem [13] and which are localized at topological charge.

In center vortex field configurations topological charge is concentrated at the intersection points and other singular points like twisting points [15]. Near these singular points we expect localization of quark zero modes which could play a similar role in the explanation of chiral symmetry breaking in the vortex picture as they do in the instanton picture. In a first attempt towards an understanding of chiral symmetry breaking in the vortex picture, in the present paper we study the quark zero modes in intersecting center vortex background fields. A more rigorous understanding of the mechanism of chiral symmetry breaking in the vortex picture is presently under investigation.

We examine zero modes of the Dirac operator in vortex backgrounds in two and four dimensions for different Euclidean space-time topologies. In sections 2 and 3 fermionic zero modes in the background of vortices in $\mathbb{R}^2$ and intersecting flat vortices in $\mathbb{R}^4$ are studied. In sections 4 and 5 this analysis is repeated for space-time given by $\mathbb{T}^2$ and $\mathbb{T}^4$, respectively.

# 2 Fermionic zero modes in non-intersecting center vortex fields

In $D = 4$ center vortices represent closed two-dimensional flux sheets. We are interested here in the quark modes in the background of such vortex sheets. Locally a center vortex sheet represents a two-dimensional plane. For simplicity we will consider in the following flat (planar) vortex sheets. In this case we have translational invariance parallel to the vortex sheets and the solution of the Dirac equation reduces to the two-dimensional problem in the plane defined by the 2 directions perpendicular to the vortex sheet. In this plane the vortex appears as intersection point. We will first study the Dirac equation in this plane, i.e. in $D = 2$.

We will consider the zero modes of the Dirac operator in the background of Abelian gauge potentials representing Dirac strings and center vortices. We can consider these gauge potentials as living in the Cartan sub-algebra of a $SU(2)$ gauge group. Having this in mind we say that a gauge potential describes a center vortex or Dirac string at a point $z_0$, if the Wilson loop around $z_0$ is $-\mathbb{1}$ or $+\mathbb{1}$, respectively. Equivalently, the magnetic flux carried by a center vortex or a Dirac string is given by $\Phi_{center} = (m + 1/2)\,,\ m \in \mathbb{Z}$ or $\Phi_{Dirac} = m\,,\ m \in \mathbb{Z}$, respectively[1].

The solution of the Dirac equation in two dimensions can be related to the theory of functions of a complex variable. We introduce the complex variable $z$ by

$$z = x + iy\,, \quad \bar{z} = x - iy\,, \quad \partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right)\,, \quad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right) \tag{2.1}$$

and a complex notation for the gauge potential. We define[1]

$$A_z := \frac{1}{2}\left(A_x - iA_y\right)\,, \quad A_{\bar{z}} := \frac{1}{2}\left(A_x + iA_y\right) = -\overline{A_z}\,, \quad A_x = 2i\Im A_z\,, \quad A_y = 2i\Re A_z\,, \tag{2.2}$$

where $\Im A_z$ is the imaginary part and $\Re A_z$ is the real part of $A_z$. The Dirac equation[1]

$$i\gamma_\mu \mathrm{D}_\mu \psi = \lambda\psi\,, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\,, \quad \mathrm{D}_\mu = \partial_\mu + A_\mu \tag{2.3}$$

in spinor components reads

$$2i\left(\partial_z + A_z\right)\psi_2 = \lambda\psi_1\,, \tag{2.4}$$

$$2i\left(\partial_{\bar{z}} + A_{\bar{z}}\right)\psi_1 = \lambda\psi_2\,. \tag{2.5}$$

[1]Our conventions are summarized in Appendix A.

The most simple case is the free Dirac equation, i.e. $A_z = 0$. Zero modes ($\lambda = 0$) of the free Dirac equation are obviously given by analytic functions $\psi_1$ and anti-analytic functions $\psi_2$. A normalizable zero mode has to go to zero at infinity. But every (anti-)analytic function without singularities which goes to zero at infinity has to be zero everywhere. This means there are no *smooth* zero modes. In principle the zero mode may have singularities, in this case poles. But a pole in the zero mode is a non-integrable singularity and the zero mode would be not normalizable. Hence, there are no normalizable zero modes for the free Dirac equation.

Next we consider the Dirac field in the background of a straight plane center vortex. Its gauge potential reads

$$A_x = -y\frac{f(r^2)}{r^2}\frac{i}{2}, \quad A_y = x\frac{f(r^2)}{r^2}\frac{i}{2}, \quad r^2 = x^2 + y^2 = z\bar{z}\,. \tag{2.6}$$

Here we introduced the profile function $f(r^2)$ which fulfills $f(0) = 0$ and $f(r^2) \to 1$ as $r \to \infty$. The complexified gauge potential is then given by

$$A_z := \frac{1}{2}\left(A_x - iA_y\right) = \frac{f(r^2)}{4r^2}\left(-iy + x\right) = \frac{f(r^2)}{4r^2}\left(\bar{z}\right)\,. \tag{2.7}$$

Introducing the function

$$\phi(x) := \int_1^x \frac{f(x')}{2x'}\mathrm{d}x' \tag{2.8}$$

the gauge potential $A_z$ can simply be written as

$$A_z = \frac{1}{2}\partial_z \phi(z\bar{z})\,. \tag{2.9}$$

Inserting the gauge potential into the Dirac equation we obtain the differential equations

$$2i\left(\partial_z + \frac{1}{2}\partial_z\phi(z\bar{z})\right)\psi_2 = \lambda\psi_1\,, \tag{2.10}$$

$$2i\left(\partial_{\bar{z}} - \frac{1}{2}\partial_{\bar{z}}\phi(z\bar{z})\right)\psi_1 = \lambda\psi_2\,. \tag{2.11}$$

We are mainly interested in the zero modes $\lambda = 0$. Furthermore, let us first consider an idealized vortex with $f(r^2) \equiv 1$ (this function obviously does not have the properties of a profile function, because $f(0) = 1 \neq 0$). For $f(r^2) = 1$ the function $\phi$ becomes $\phi(r^2) = \frac{1}{2}\log(r^2)$, which is - up to a factor - the Greens function of the Laplace operator in two dimensions. For $\lambda = 0$ and $\phi(r^2) = \frac{1}{2}\log(r^2)$ the differential equations (2.10, 2.11) can again be easily solved

$$\psi_1 = \left(\sqrt[4]{z\bar{z}}\right)\chi_1(z)\,, \tag{2.12}$$

$$\psi_2 = \left(\sqrt[4]{z\bar{z}}\right)^{-1}\overline{\chi_2(z)}\,, \tag{2.13}$$

where $\chi_1$ and $\chi_2$ are analytic functions of $z$ and $\overline{\chi_2(z)}$ is the complex conjugate of $\chi_2(z)$. Choosing e.g. $\chi_1 = 1$ we get $\psi_1 = \sqrt{r}$, i.e. a real-valued function of $x$ and $y$. The zero modes (2.12, 2.13) of the Dirac operator in the background of a single center vortex are not normalizable. This is because normalizable analytic functions $\chi_{1/2}$ have to approach zero at infinity, and therefore, have to be identically zero or have a pole which yields a non-integrable singularity as was already discussed above.

This result is in accord with the index theorem for $U(1)$ gauge fields in $D = 2$. The theorem states that the difference between the numbers of right- and left-handed fermionic zero modes, say $n$, in the background of a $U(1)$ gauge potential on $\mathbb{R}^2$ is related to the total flux $\Phi = \frac{1}{2\pi i}\int_{\mathbb{R}^2} F$ by [16]

$$n = [\Phi] \, , \tag{2.14}$$

where $[x]$ is the largest integer smaller than $x \in \mathbb{R}$ (i.e. $[1] = 0$). In the case of a single center vortex on $\mathbb{R}^2$ the total flux is equal to $1/2$, i.e. the number of left-handed zero modes is equal to the number of right-handed zero modes (in the present case there are no normalizable left- or right-handed zero modes).

To get rid of the singularity of the vortex field at $z = 0$ we smear out the vortex with a profile function $f(r^2) = r^2/(\varepsilon^2 + r^2)$, $\varepsilon \in \mathbb{R}$, i.e. we work with the gauge potential

$$A_x = -\frac{y}{r^2+\varepsilon^2}\frac{i}{2}\,, \quad A_y = \frac{x}{r^2+\varepsilon^2}\frac{i}{2}\,. \tag{2.15}$$

For this profile function $f$ we obtain $\phi(r^2) = \frac{1}{2}\log(r^2 + \varepsilon^2)$. For $\lambda = 0$ the differential equations (2.10, 2.11) can be solved also in this case:

$$\psi_1 = \left(\sqrt[4]{r^2+\varepsilon^2}\right)\tilde{\chi}_1(z)\,, \tag{2.16}$$

$$\psi_2 = \left(\sqrt[4]{r^2+\varepsilon^2}\right)^{-1}\overline{\tilde{\chi}_2(z)}\,. \tag{2.17}$$

For $\tilde{\chi}_{1/2} = 1$ these solutions are obviously not normalizable. With the same arguments as before we conclude that there are no normalizable zero modes.

As discussed already above in connection with the index theorem the asymptotic behavior of the zero mode of the Dirac operator changes if we change the magnitude of the magnetic flux. Multiplying the gauge potential $A$ in (2.15) by a factor $\rho \in \mathbb{R}^+$ the flux $\Phi$ becomes $\rho/2$ and the solutions $\psi_{1/2}$, cf. eqs. (2.16, 2.17), change to $(\psi_{1/2})^\rho$. This means that for $\rho > 2$ we get a total flux $\Phi > 1$ and a normalizable zero mode with chirality $-1$:

$$\psi_2 = (\sqrt[4]{r^2+\varepsilon^2})^{-\rho}\,, \quad \psi_1 \equiv 0\,. \tag{2.18}$$

The probability density of this zero mode has a maximum at $r = 0$, i.e. at the location of the vortex.

Since the differential equations (2.10, 2.11) are of first order type it is also possible to explicitly write down zero modes for multi-vortex background fields, because the multi-vortex gauge potential can be written as the sum over the gauge potentials of several simple vortices. The solution of the corresponding multi-vortex differential equation is then simply the product of the solutions to the several one-vortex differential equations.

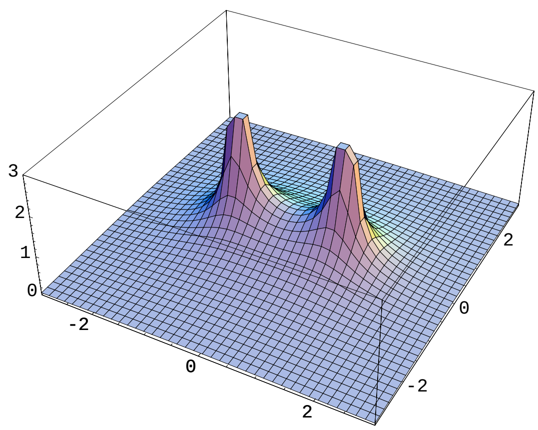


Figure 1: *Probability density of the zero mode in the background of two center vortices in $D = 2$ (see eqs. (2.19, 2.20)) for $\varepsilon = 0.01$, $a = 1$ and $b = -1$.*

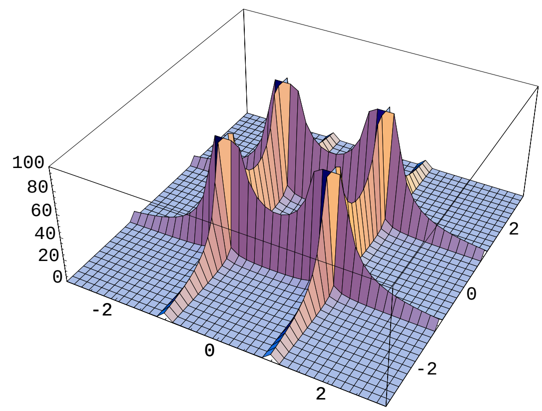


Figure 2: *Probability density of the zero mode in the background of four intersecting vortex sheets is shown in the subspace $x_4 = x_2 = 0$ for $a = b = 1$ and $\varepsilon = 0.01$.*

As an example take two vortices, one at $z = a$ and a second at $z = b$. The solutions $\psi_{1/2}$ then read

$$\psi_1 = \left(\sqrt[4]{(z-a)(\bar{z}-\bar{a})+\varepsilon^2}\,\sqrt[4]{(z-b)(\bar{z}-\bar{b})+\varepsilon^2}\right)\chi_1(z)\,, \tag{2.19}$$

$$\psi_2 = \left(\sqrt[4]{(z-a)(\bar{z}-\bar{a})+\varepsilon^2}\,\sqrt[4]{(z-b)(\bar{z}-\bar{b})+\varepsilon^2}\right)^{-1}\overline{\chi_2(z)}\,. \tag{2.20}$$

There is one quasi-normalizable[2] zero mode: $(0, \psi_2)$. This zero mode is obviously localized at the centers of the vortices (at $z = a$ and $z = b$) and on the line between them, cf. fig. 1. A similar calculation on the compact sphere $S^2$ yields a normalizable zero mode, cf. Appendix B. According to the index theorem we obtain normalizable zero modes as soon as the net flux of the vortices exceeds 1 (i.e. two units of the flux of a center vortex).

# 3 Fermionic zero modes of the Dirac operator for intersecting center vortex fields

So far we have considered the fermionic modes in the background of parallel non-intersecting flat center vortex sheets. Due to the translational invariance parallel to the vortex sheets it was sufficient to study the Dirac operator in the plane perpendicular to the vortex sheet, where the vortices appear as intersection points. In the confined phase the center vortices percolate [17]. The vortices then have arbitrary directions and also intersect. In intersection points all four space directions participate and, consequently to study the fermionic

[2]Strictly speaking we have to increase the magnetic flux infinitesimally to get a normalizable zero mode.

modes in the background of intersecting vortices we have to consider the full 4-dimensional Dirac operator.

In the following we consider four vortex sheets consisting of two orthogonal pairs of parallel vortex sheets. One vortex pair is given by two planes parallel to the $x_1 - x_2$ plane located at $x_4 = 0$ and $x_3 = \pm a$. The other vortex pair consists of two planes parallel to the $x_3 - x_4$ plane located at $x_2 = 0$ and $x_1 = \pm b$. The four vortices intersect in four points given by $x_4 = x_2 = 0$, $x_3 = \pm a$, $x_1 = \pm b$. Each of the intersection points carries topological charge $\pm 1/2$. If we choose the orientation of the flux of the two vortex sheets to be equal we find a total Pontryagin index of $\nu = \pm 2$, otherwise (if the fluxes are anti-parallelly oriented) the Pontryagin index vanishes. For definiteness we choose the direction of the flux in parallel vortex pairs to be the equal. Then the gauge potential can be chosen as

$$\begin{aligned} A_1 &= \left[-x_2 \frac{f(s_+^2)}{s_+^2} - x_2 \frac{f(s_-^2)}{s_-^2}\right] \frac{i}{2}, \quad A_2 = \left[(x_1+b)\frac{f(s_+^2)}{s_+^2} + (x_1-b)\frac{f(s_-^2)}{s_-^2}\right] \frac{i}{2}, \\ A_3 &= \left[-x_4 \frac{f(r_+^2)}{r_+^2} - x_4 \frac{f(r_-^2)}{r_-^2}\right] \frac{i}{2}, \quad A_4 = \left[(x_3+a)\frac{f(r_+^2)}{r_+^2} + (x_3-a)\frac{f(r_-^2)}{r_-^2}\right] \frac{i}{2}, \end{aligned}$$

where

$$r_\pm^2 = (x_3 \pm a)^2 + x_4^2\,, \quad s_\pm^2 = (x_1 \pm b)^2 + x_2^2\,, \quad a, b \in \mathbb{R}^+\,, \tag{3.21}$$

$$f(r^2) = r^2/(r^2 + \varepsilon^2)\,. \tag{3.22}$$

The field strength $F_{\mu\nu}$ of the above gauge potential is given by

$$\begin{aligned} F_{12} &= \frac{i}{2}\left(\frac{2\varepsilon^2}{(s_+^2+\varepsilon^2)^2} + \frac{2\varepsilon^2}{(s_-^2+\varepsilon^2)^2}\right) \\ F_{34} &= \frac{i}{2}\left(\frac{2\varepsilon^2}{(r_+^2+\varepsilon^2)^2} + \frac{2\varepsilon^2}{(r_-^2+\varepsilon^2)^2}\right). \end{aligned}$$

Splitting the field strength into (anti-)self-dual components

$$F_{\mu\nu} = \left(\alpha \eta_{\mu\nu}^3 + \beta \bar{\eta}_{\mu\nu}^3\right)\,, \tag{3.23}$$

where $\eta_{\mu\nu}^a$ and $\bar{\eta}_{\mu\nu}^a$ are the t' Hooft symbols and $\alpha = F_{12} + F_{34}$, $\beta = F_{12} - F_{34}$, one finds that in the limit $\varepsilon \to 0$ the field strength is self-dual ($\beta = 0$) at the intersection points.

To determine the zero modes of the Dirac operator in the background of the considered intersecting vortices we introduce complex variables $u$ and $v$ [18]

$$u = x_3 + i x_4\,, \quad v = x_1 + i x_2 \tag{3.24}$$

and the corresponding complex derivatives $\partial_u$ and $\partial_v$

$$\partial_u = \frac{1}{2}\left(\partial_{x_3} - i\partial_{x_4}\right)\,, \quad \partial_v = \frac{1}{2}\left(\partial_{x_1} - i\partial_{x_2}\right)\,. \tag{3.25}$$

We also introduce two complexified gauge potentials $A_u$ and $A_v$ as in eq. (2.7). These gauge potentials can be written as

$$A_u \;=\tfrac{1}{2}\partial_u\phi\,, \quad A_v \;=\frac{1}{2}\partial_v\phi\,, \tag{3.26}$$

where

$$\begin{aligned}\phi=\phi(u,\bar u,v,\bar v) \;&=\log\Big[\;((u-a)(\bar u-a)+\varepsilon^2)((u+a)(\bar u+a)+\varepsilon^2)\\ &\qquad ((v-b)(\bar u-b)+\varepsilon^2)((u+b)(\bar u+b)+\varepsilon^2)\Big]\,.\end{aligned} \tag{3.27}$$

The Dirac operator in the background of the gauge potential (3.21) reads

$$i\gamma_\mu \mathrm{D}_\mu\psi \;=\; i\begin{pmatrix} 0 & 0 & i\mathrm{D}_4+\mathrm{D}_3 & \mathrm{D}_1-i\mathrm{D}_2\\ 0 & 0 & \mathrm{D}_1+i\mathrm{D}_2 & i\mathrm{D}_4-\mathrm{D}_3\\ -i\mathrm{D}_4+\mathrm{D}_3 & \mathrm{D}_1-i\mathrm{D}_2 & 0 & 0\\ \mathrm{D}_1+i\mathrm{D}_2 & -i\mathrm{D}_4-\mathrm{D}_3 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}, \tag{3.28}$$

where $\mathrm{D}_\mu=\partial_\mu+A_\mu$ is the covariant derivative. For the considered vortex gauge potential (3.21) the eigenvalue equation (2.3) for the Dirac operator reduces to four coupled differential equations:

$$\begin{aligned}\lambda\psi_1 &= i\left(2\partial_{\bar u}-\frac{1}{2}\partial_{\bar u}\phi\right)\psi_3+i\left(2\partial_v+\frac{1}{2}\partial_v\phi\right)\psi_4\,,\\ \lambda\psi_2 &= i\left(2\partial_{\bar v}-\frac{1}{2}\partial_{\bar v}\phi\right)\psi_3+i\left(-2\partial_u-\frac{1}{2}\partial_u\phi\right)\psi_4\,,\\ \lambda\psi_3 &= i\left(2\partial_u+\frac{1}{2}\partial_u\phi\right)\psi_1+i\left(2\partial_v+\frac{1}{2}\partial_v\phi\right)\psi_2\,,\\ \lambda\psi_4 &= i\left(2\partial_{\bar v}-\frac{1}{2}\partial_{\bar v}\phi\right)\psi_1+i\left(-2\partial_{\bar u}+\frac{1}{2}\partial_{\bar u}\phi\right)\psi_2\,,\end{aligned}$$

which can be easily solved for $\lambda=0$, since in this case the upper and lower components of the Dirac spinor decouple. The $\lambda=0$ solutions read

$$\begin{aligned}\psi_1 &= \left(\sqrt[4]{(r_+^2+\varepsilon^2)(r_-^2+\varepsilon^2)}\right)^{-1}\sqrt[4]{(s_+^2+\varepsilon^2)(s_-^2+\varepsilon^2)}\chi_1(\bar u,v)\,,\\ \psi_2 &= \sqrt[4]{(r_+^2+\varepsilon^2)(r_-^2+\varepsilon^2)}\left(\sqrt[4]{(s_+^2+\varepsilon^2)(s_-^2+\varepsilon^2)}\right)^{-1}\chi_2(u,\bar v)\,,\\ \psi_3 &= \sqrt[4]{(r_+^2+\varepsilon^2)(r_-^2+\varepsilon^2)}\sqrt[4]{(s_+^2+\varepsilon^2)(s_-^2+\varepsilon^2)}\chi_3(u,v)\,,\\ \psi_4 &= \left(\sqrt[4]{(r_+^2+\varepsilon^2)(r_-^2+\varepsilon^2)}\right)^{-1}\left(\sqrt[4]{(s_+^2+\varepsilon^2)(s_-^2+\varepsilon^2)}\right)^{-1}\chi_4(\bar u,\bar v)\,.\end{aligned}$$

The analytic functions $\chi_i$ have to be chosen constant to avoid non-integrable singularities as was discussed for the $D=2$ case. The only spinor component going to zero at infinity is

$\psi_4$. Therefore, only the spinor with non-vanishing component $\psi_4$ (and all other components zero) yields a normalizable zero mode (normalizable up to a logarithmic divergence from the integration $r_\pm \to \infty$ and $s_\pm \to \infty$ - but these divergences become integrable if the vortex gauge potential is multiplied with $\rho = 1 + \alpha$, $\alpha > 0$). Embedding the $U(1)$ gauge group into an $SU(2)$ gauge group results in a second zero mode with opposite iso-spin (corresponding to inverting the sign in front of the vortex gauge potential (3.21)) and the same chirality as the above zero mode. This second zero mode is given by[3]

$$\begin{aligned}
\psi_1 &= \psi_2 = \psi_4 \equiv 0\,, \\
\psi_3 &= \left(\sqrt[4]{(r_+^2+\varepsilon^2)(r_-^2+\varepsilon^2)}\right)^{-1}\left(\sqrt[4]{(s_+^2+\varepsilon^2)(s_-^2+\varepsilon^2)}\right)^{-1}.
\end{aligned}$$

The probability density of the zero modes is peaked at the four vortex intersection points and at the vortex sheets, cf. fig. 2.

# 4 Center vortices and Dirac equation on the 2-torus

In this chapter we consider zero modes of the Dirac operator in the background of Abelian gauge potentials representing Dirac strings and center vortices on the torus $\mathbb{T}^2$. There is a variety of reasons for studying $\mathbb{T}^2$ in addition to $\mathbb{R}^2$: first, $\mathbb{T}^2$ allows to use the Atiyah-Singer index theorem in a stringent fashion and allows for normalizable spinors in the background of integer flux. Second, from a physical point of view one would want the mechanism of chiral symmetry breaking to depend on local quantities rather than on global characteristics like boundary conditions imposed from the manifold. Therefore the proposed mechanism has to be checked for different topologies of the space-time manifold. Third, the torus simulates a periodic arrangement of vortices. This is much closer to a percolated vortex cluster than a single vortex in $\mathbb{R}^2$. It will turn out that the zero modes are again localized at the position of the vortex, thus strengthening the point of view that the zero modes are indeed influenced by local properties of the gauge potential only. Finally, the torus is the space-time manifold that is also used in lattice calculations.

## 4.1 Periodicity properties of the gauge potential on the torus

We first consider flat vortices so that we can restrict ourselves to a two-torus where the vortices appear as piercing points. Instead of working on the compact torus we can work on its covering manifold - on $\tilde{\mathbb{T}}^2 = \mathbb{R}^2 \equiv \mathbb{C}$. But in this case we have to demand that physical, i.e. gauge invariant, observables are periodic. This implies that the gauge potential has to be periodic up to gauge transformations. We choose the two-torus to have circumferences 1 and $\tau$. Then the gauge potential $A$ has to fulfill "quasi-periodicity" conditions

$$A(z+1) = A^{U_x(z)}(z)\,, \quad A(z+i\tau) = A^{U_y(z)}(z)\,, \tag{4.29}$$

[3] Embedding the $U(1)$ gauge group into $SU(2)$ gives the Pontryagin index 2 for this configuration. This corresponds to the 2 observed zero modes.

where $A_\mu^U = U^{-1}A_\mu U + U^{-1}\partial_\mu U$ is the gauge transform of $A$. The transition functions $U_x, U_y$ have to fulfill the cocycle condition

$$U_x(z)U_y(z+1) = U_y(z)U_x(z+i\tau)\,. \tag{4.30}$$

We can always choose $U_x = \mathbb{1}$ and $U_y$ independent of $y$. This implies periodicity of $A(z)$ in $x$-direction. Furthermore, $U_y$ as function of $x$ defines a mapping from $S^1$ into $U(1)$. Such mappings fall into homotopy classes $\pi_1(U(1)) = \pi_1(S^1)$ which are characterized by a winding number $n$. A simple calculation shows that this winding number is related to the magnetic flux $\Phi$ through the torus

$$\Phi = -n\,, \tag{4.31}$$

reflecting the quantization of the magnetic flux through the torus.

## 4.2 Dirac string on the torus

Before we write down the gauge potential of center vortices on the torus let us consider the gauge potential of a single Dirac string. The reason is that a single center vortex does not exist on a torus while a single Dirac string does.

On the two-torus the gauge potential of a singular point-like object can be expressed by means of the theta functions

$$\theta(z, i\tau) = \sum_{n\in\mathbb{Z}} e^{-\pi\tau n^2 + 2\pi i n z}\,, \quad \tau \in \mathbb{R}_+ \tag{4.32}$$

which are analytic in $z$ and obey the periodicity properties

$$\theta(z+1, i\tau) = \theta(z, i\tau)\,, \quad \theta(z+i\tau, i\tau) = e^{\pi\tau - 2\pi i z}\theta(z, i\tau)\,. \tag{4.33}$$

The only zeros of this function are at the points [19]

$$z = (m+1/2) + (n+1/2)i\tau\,, \quad m, n \in \mathbb{Z}\,. \tag{4.34}$$

For subsequent consideration let us also introduce the real-valued function

$$\phi_0(z,\bar{z}) = \frac{1}{2}\log\left(\theta(z+1/2+1/2i\tau - z_0, i\tau)\overline{\theta(z+1/2+1/2i\tau - z_0, i\tau)}\right), \tag{4.35}$$

which (up to a factor of $2\pi$) represents a (non-periodic) Greens function of the Laplacian on the two-torus[4]. By Taylor expanding $\theta(z+1/2+1/2i\tau - z_0, i\tau)$ around its zero at $z_0$ one finds that the Greens function $\phi_0(z,\bar{z})$ behaves near $z_0$ as $\log|z-z_0|$. By means of this Greens function the gauge potential of a Dirac string on the two-torus can be expressed as

$$A_x = -i\partial_y\phi_0\,, \quad A_y = i\partial_x\phi_0\,. \tag{4.36}$$

[4]On the torus there is no periodic Greens function of the Laplacian

Using the same notation as in eq. (2.2)

$$A_z := 1/2(A_x - iA_y)\,, \quad A_x = 2i\Im(A_z)\,, \quad A_y = 2i\Re(A_z)\,, \tag{4.37}$$

we obtain

$$A_z = \frac{1}{2}(-i\partial_y\phi_0 + \partial_x\phi_0) = \partial_z\phi_0 = \frac{1}{2}\frac{\partial_z\theta_0}{\theta_0}\,, \tag{4.38}$$

where we have introduced the abbreviation $\theta_0(z) = \theta(z+1/2+1/2i\tau - z_0, i\tau)$ and used that $\theta_0$ is an analytic function of $z$ and $\bar\theta_0$ is an anti-analytic function, i.e. it does not depend on $z$. The periodicity properties of $A_z, A_x$ and $A_y$ can simply be derived using equation (4.33)

$$\begin{aligned} A_z(z+1) &= A_z(z)\,, \quad A_z(z+i\tau) = -\pi i + A_z(z)\,, & (4.39)\\ A_y(z+1) &= A_y(z+i\tau) = A_y(z)\,, & (4.40)\\ A_x(z+1) &= A_x(z)\,, \quad A_x(z+i\tau) = -2\pi i + A_x(z)\,. & (4.41) \end{aligned}$$

This means that the gauge potential fulfills equation (4.29) with $U_x(x,y) = \mathbb{1}$ and $U_y(x,y) = \exp(-2\pi i x)\mathbb{1}$. Furthermore, computing explicitly the flux going through the torus, we obtain (using Stokes theorem)

$$\begin{aligned} \Phi &= \frac{1}{2\pi i}\left(\int_{(0,0)}^{(1,0)} A_x \mathrm{d}x + \int_{(1,0)}^{(1,\tau)} A_y \mathrm{d}y + \int_{(1,\tau)}^{(0,\tau)} A_x \mathrm{d}x + \int_{(0,\tau)}^{(0,0)} A_y \mathrm{d}y\right)\\ &= \frac{1}{2\pi i}\left(\int_0^1 (A_x(x,0) - A_x(x,\tau))\mathrm{d}x + \int_0^\tau (A_y(1,y) - A_y(0,y))\mathrm{d}y\right) = 1\,, \end{aligned} \tag{4.42}$$

which is consistent with (4.31). The Dirac string is located at the point where $\theta_0(z) = 0$, i.e. at $z = z_0 + m + in\tau\,,\ m,n \in \mathbb{Z}$. The field strength of the Dirac string configuration vanishes at points where $\theta_0(z) \neq 0$, because

$$F_{xy} = \partial_x A_y - \partial_y A_x = i(\partial_x^2 + \partial_y^2)\phi = \Re(4\partial_{\bar z}\partial_z\phi) = 2\Re\partial_{\bar z}(\partial_z\theta_0/\theta_0) = 0\,. \tag{4.43}$$

Here we used again that $\theta_0$ is an analytic function, i.e. it is independent of $\bar z$. On the other hand the flux through the torus is 1, cf. (4.42), from which we conclude that we have a Dirac string (which is represented by a point in $D = 2$) at the zero of the function $\theta_0(z)$, i.e. at $z = z_0 + m + ni\tau$.

The Dirac string can also be written as a pure gauge. If we define the $U(1)$ gauge function

$$g(z) = \frac{\theta_0(z)}{|\theta_0(z)|} = \sqrt{\frac{\theta_0(z)}{\overline{\theta_0(z)}}} \in U(1)\,, \tag{4.44}$$

which is singular at the zeros of $\theta_0(z)$, the gauge potential

$$A_\mu = g^{-1}\partial_\mu g \tag{4.45}$$

becomes

$$A_z = \frac{1}{2}(A_x - iA_y) = g^{-1}\partial_z g = \sqrt{\frac{\overline{\theta_0(z)}}{\theta_0(z)}}\partial_z\sqrt{\frac{\theta_0(z)}{\overline{\theta_0(z)}}} \tag{4.46}$$

$$= \frac{1}{2}\frac{\partial_z\theta_0(z)}{\theta_0(z)}, \tag{4.47}$$

which agrees with eq. (4.38). The periodicity properties of $g$ are as follows:

$$g(z+1) = g(z), \quad g(z+i\tau) = -e^{-\pi i((z-z_0)+(\bar{z}-\bar{z}_0))}g(z). \tag{4.48}$$

Since the gauge potential of the Dirac string is a pure gauge we can simply write down the zero modes of the corresponding Dirac operator. The zero mode is nothing but the gauge transformation of a constant Dirac field.

## 4.3 Fermionic zero modes of center vortices on the torus

An ideal center vortex living in the Cartan sub-algebra can be represented by half the gauge potential of a Dirac string. However, for a single center vortex it is not possible to relate the gauge potential at $z$ with the gauge potential at $z+i\tau$ by a transition function $U_y$ which is periodic in $x$. Instead we need an even number of center vortices on the torus, in accord with the quantization of magnetic flux through the torus, cf. (4.31).

We consider the configuration

$$A_z = \frac{1}{4}\left(\partial_z\theta_1/\theta_1 + \partial_z\theta_2/\theta_2\right), \quad \theta_k(z) = \theta(z+1/2+i\tau/2-z_k, i\tau), \quad k=1,2, \tag{4.49}$$

which consists of two center vortices at the points $z_1$ and $z_2$ and fulfills the periodicity properties (4.29) with transition functions $U_x = \mathbb{1}$ and $U_y = \exp(-2\pi i x)\mathbb{1} = \exp(-\pi i(\bar{z}+z))\mathbb{1}$.

We are interested in the $\lambda = 0$ eigenfunctions of the Dirac operator in the background of these two vortices. In accord with the boundary conditions to the gauge field (4.29) we impose the boundary conditions

$$\psi(z+1) = U_x(z)^{-1}\psi(z) = \psi(z), \quad \psi(z+i\tau) = U_y(z)^{-1}\psi(z) = e^{\pi i(\bar{z}+z)}\psi(z) \tag{4.50}$$

to the Dirac spinors $\psi$. The Dirac eigenvalue equation for the two spinor components reads

$$i(2\partial_z + 2A_z)\psi_2 = \lambda\psi_1, \tag{4.51}$$

$$i(2\partial_{\bar{z}} + 2A_{\bar{z}})\psi_1 = \lambda\psi_2, \tag{4.52}$$

where $A_{\bar{z}} = 1/2(A_x + iA_y) = -\overline{A_z}$. Because of the simple form of the two-vortex gauge potential (4.49) the zero modes of the Dirac operator

$$0 = (\partial_z + A_z)\psi_2 = (\partial_z + \frac{1}{4}\partial_z\theta_1/\theta_1 + \frac{1}{4}\partial_z\theta_2/\theta_2)\psi_2, \tag{4.53}$$

$$0 = (\partial_{\bar{z}} + A_{\bar{z}})\psi_1 = (\partial_{\bar{z}} - \frac{1}{4}\partial_{\bar{z}}\theta_1/\theta_1 - \frac{1}{4}\partial_{\bar{z}}\theta_2/\theta_2)\psi_1, \tag{4.54}$$

can be found explicitly. A short calculation yields

$$\psi_1 = \left(\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2\right)^{+1/4}\chi_1(z)\,, \tag{4.55}$$

$$\psi_2 = \left(\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2\right)^{-1/4}\overline{\chi_2(z)}\,. \tag{4.56}$$

The analytic functions $\chi_1(z)$ and $\chi_2(z)$ have to be chosen such that $\psi_{1/2}$ fulfill the periodicity properties (4.50). Inserting equations (4.56, 4.55) into (4.50) and using the periodicity properties (4.33) of the theta function we arrive at

$$\chi_1(z+i\tau) = e^{\left(-\pi\tau+2\pi i\left(z+\frac{i}{2}\tau-\frac{i}{2}\Im(z_1+z_2)\right)\right)}\chi_1(z)\,, \quad \chi_1(z+1)=\chi_1(z)\,, \tag{4.57}$$

$$\chi_2(z+i\tau) = e^{\left(\pi\tau-2\pi i\left(z+\frac{i}{2}\tau-\frac{i}{2}\Im(z_1+z_2)\right)\right)}\chi_2(z)\,, \quad \chi_2(z+1)=\chi_2(z)\,. \tag{4.58}$$

If we require analyticity of $\chi_2$ on the whole torus then this function is fixed (up to a factor) by the periodicity properties (4.58) to be the theta function

$$\chi_2(z) = \theta(z+\frac{i}{2}\tau-\frac{i}{2}\Im(z_1+z_2), i\tau)\,. \tag{4.59}$$

This is proven in Appendix C. The function $\psi_2$, cf. (4.56), is obviously normalizable, because the singularities at $z_1$ and $z_2$ are integrable. The only zeros of $\chi_2(z)$ are at the points $z=\frac{1}{2}+\frac{i}{2}\Im(z_1+z_2)+m+ni\tau$. On the other hand the required periodicity properties for $\chi_1$ (4.57) show that there is no (non-trivial) function $\chi_1$ which is analytic on the whole torus. The function $\chi_1$ has to have at least one pole. Such a solution is given by the inverse of a theta function

$$\chi_1(z) = 1/\theta(z+\frac{i}{2}\tau-\frac{i}{2}\Im(z_1+z_2), i\tau)\,. \tag{4.60}$$

The poles of this function are at the points $z=z_{mn}=\frac{1}{2}+\frac{i}{2}\Im(z_1+z_2)+m+ni\tau\,,\ m,n\in\mathbb{Z}$. Our considerations show that there is only one normalizable zero mode (with $\psi_1\equiv 0$). If the component $\psi_1$ of the Dirac spinor is not identically zero then it would have a pole at $z=z_{mn}$ and this pole would yield a logarithmic divergence.

The vortex configuration (4.49) is singular at the positions of the vortices. To get rid off these singularities one can smear out the vortices. This is done in Appendix D, where we also present the corresponding fermionic zero mode. Furthermore, in Appendix E multi-vortex configurations and the corresponding zero modes of the Dirac operator are presented.

There is another interesting point related to the periodicity properties given by (4.29, 4.50). Multiplying the transition functions $U_x$ and $U_y$ by constant phases, say $e^{-i\alpha}$ and $e^{-i\beta}$, resp., the center vortex and Dirac string gauge potentials, (4.49) and (4.36), still fulfill the periodicity properties with the new transition functions $\tilde{U}_x=e^{-i\alpha}\mathbb{1}$ and $\tilde{U}_y=e^{-2\pi ix-i\beta}\mathbb{1}$. But these new periodicity properties change the zero modes of the Dirac operator by changing the periodicity properties of the analytic functions $\chi_{1/2}(z)$. The new solution reads

$$\tilde{\chi}_2(z) = e^{-i\alpha z}\theta(z+\frac{i}{2}\tau-\frac{i\alpha}{2\pi}\tau-\frac{i}{2}\Im(z_1+z_2)+\frac{\beta}{2\pi}, i\tau) \tag{4.61}$$

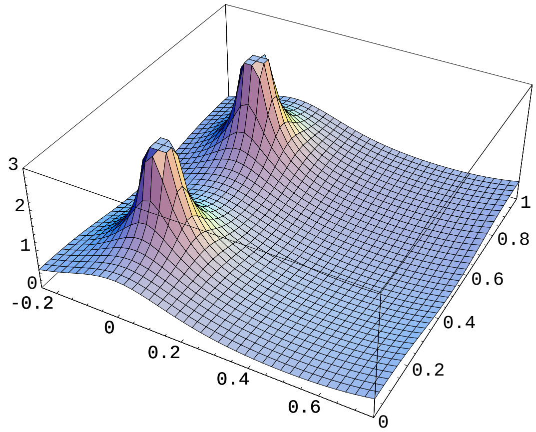


Figure 3: *Probability density of the zero mode in the background of two center vortices on* $\mathbb{T}^2$ *for* $\tau = 1$, $\varepsilon = 0.01$, $z_1 = 0.25i$ *and* $z_2 = 0.75i$.

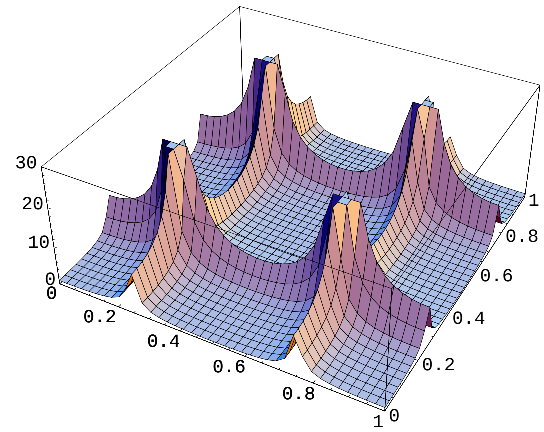


Figure 4: *Probability density of the zero mode in the background of four intersecting vortex sheets on* $\mathbb{T}^4$ *shown in the subspace* $x_1 = x_3 = 0$ *for* $u_1 = v_1 = 0.25i$ , $u_2 = v_2 = 0.75i$ *and* $\varepsilon = 0.01$.

The zeros of this function are at the points $z = \frac{1}{2} + \frac{i}{2}\Im(z_1 + z_2) + \frac{i\alpha}{2\pi}\tau - \frac{\beta}{2\pi} + m + ni\tau$, i.e. the zeros are shifted by $\frac{i\alpha\tau - \beta}{2\pi}$ compared to the original case, where $\tilde{U}_x = \mathbb{1}$ and $\tilde{U}_y = e^{-2\pi i x}\mathbb{1}$.

The change of the transition functions by multiplying with constant phases $e^{-i\alpha}$ and $e^{-i\beta}$ is equivalent to introducing a constant background gauge potential $A_z = \frac{1}{2}(\frac{\beta}{\tau} + i\alpha)$ and leaving the transition functions unchanged.

The above considerations have shown that for a two-vortex gauge potential (smeared out or not) we get exactly one normalizable zero mode which has exactly one zero on the torus. The position of the zero depends on the imaginary parts (y-coordinates) of the positions of the vortices and on the periodicity properties of the gauge potential (or equivalently on the presence of a constant background field) and of the spinor field. Furthermore, the probability density of the spinor field is peaked at the positions of the vortices.

# 5 Fermionic zero modes for intersecting vortices on the 4-torus

As in the case of space time $\mathbb{R}^4$ the fermionic zero modes for intersecting vortices on $\mathbb{T}^4$ can be explicitly written down.

We consider four smeared out center vortex sheets consisting of two orthogonal pairs of parallel vortex sheets intersecting in 4 points as in section 3. Introducing complex variables $u$ and $v$, cf. eq. (3.25), the complexified gauge potential $A_z$ can be chosen as

$$A_z \quad = \quad \partial_u(\phi(u, u_1) + \phi(u, u_2)) + \partial_v(\phi(v, v_1) + \phi(v, v_2))\,,$$

where the function $\phi(z, z_k)$ is defined in eq. (D.72). As in the case of the space-time manifold $\mathbb{R}^4$ there is only one normalizable zero mode given by

$$\begin{aligned}
\psi_1 &= \psi_2 = \psi_3 \equiv 0\,, \\
\psi_4 &= \left(\prod_{k=1}^{2}\left(\theta^+(u,u_k)\overline{\theta^+(u,u_k)} + \theta^-(u,u_k)\overline{\theta^-(u,u_k)}\right)\right)^{-1/4} \times \\
&\quad \times \left(\prod_{k=1}^{2}\left(\theta^+(v,v_k)\overline{\theta^+(v,v_k)} + \theta^-(v,v_k)\overline{\theta^-(v,v_k)}\right)\right)^{-1/4} \overline{\chi_4(u,v)}\,,
\end{aligned}$$

where $\chi_4(u,v)$ is an analytic function of $u$ and $v$ given by

$$\chi_4(u,v) = \theta(u + \frac{i}{2}\tau - \frac{i}{2}\Im(u_1+u_2), i\tau)\theta(v + \frac{i}{2}\tau - \frac{i}{2}\Im(v_1+v_2), i\tau)\,. \tag{5.62}$$

The probability distribution of this zero mode in the plane $\Re u = \Re v = 0$ for $u_1 = u_2 = 0.25i$, $u_2 = v_2 = 0.75i$ is plotted in fig. 4.

# 6 Concluding remarks

In the present paper we have studied the properties of fermionic zero modes in a center vortex background field. We have demonstrated that these zero modes are concentrated at the localization of the center vortices. In accord with this the probability density of these zero modes is sharply peaked at the vortex intersection points which carry (localized) topological charge 1/2. This result is consistent with the localization of the fermionic zero modes in an instanton background field at the instanton center. In fact lattice calculations show a strong correlation between the topological charge density distribution and the distribution of the quark condensate $< \bar{q}(x)q(x) >$, the order parameter of chiral symmetry breaking. Given the localization of the quark zero modes at the localization of topological charge, we expect the quark zero modes in the vortex background field to play a crucial role for the spontaneous breaking of chiral symmetry in the vortex picture. This will be subject to future investigations.

# 7 Acknowledgments

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# A Conventions

We choose the generators of the gauge group to be anti-hermitian. Therefore the components $A_\mu$ of the gauge potential are anti-hermitian, e.g. purely imaginary for the gauge group $U(1)$. The magnetic flux $\Phi$ through a closed loop $\mathcal{C}$ is defined by

$$\Phi = \frac{1}{2\pi i} \oint_{\mathcal{C}} A_\mu \mathrm{d}x_\mu \tag{A.63}$$

and thus real valued.

The complex conjugate of the complex number z is denoted by $\bar{z}$. Furthermore, $\Re(z)$ and $\Im(z)$ denote real and the imaginary part of $z$, respectively.

We consider the Dirac equation in Euclidean space-time. In $D = 2$ we choose the $2 \times 2$ Dirac matrices

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_5 = -i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.64}$$

In $D = 4$ we use the chiral representation for the Dirac matrices:

$$\gamma_4 = \begin{pmatrix} 0 & i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad \gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{A.65}$$

where $\mathbb{1}$ is the $2 \times 2$ unit matrix and $\sigma_i$ are the (hermitian) Pauli matrices.

# B Dirac zero modes on $S^2$

We consider the Dirac operator on the sphere $S^2$ with radius $R$. We use stereographic coordinates $x_{1/2}$ on $S^2$ defined by

$$y_i = \frac{2R^2}{R^2 + x^2} x_i, \quad i = 1, 2, \quad y_3 = \frac{R^2 - x^2}{R^2 + x^2} R, \quad x^2 = x_1^2 + x_2^2, \tag{B.66}$$

where $\vec{y} = (y_1, y_2, y_3)$ is the vector in $\mathbb{R}^3$ of the corresponding point of the embedded sphere $S^2$ with radius $R$. The metric on $S^2$ in stereographic coordinates has the form

$$\mathrm{d}s^2 = \Omega_R^2 (\mathrm{d}x_1^2 + \mathrm{d}x_2^2), \tag{B.67}$$

where $\Omega_R = 2R^2/(R^2 + x^2)$. The Dirac operator $\hat{\mathrm{D}}$ in these coordinates is given by [21]

$$\hat{\mathrm{D}} = \Omega_R^{-3/2} \mathrm{D} \Omega_R^{1/2}, \tag{B.68}$$

where D is the Dirac operator on $\mathbb{R}^2$. Therefore the zero modes $\hat{\psi}$ of the Dirac operator on $S^2$ are related to the zero modes $\psi$ of the Dirac operator on $\mathbb{R}^2$ by

$$\hat{\psi} = \Omega_R^{-1/2} \psi. \tag{B.69}$$

With $z = x_1 + ix_2$ the zero mode of the Dirac operator in the presence of two smeared out center vortices at the points $z = a$ and $z = b$ reads

$$\psi_1 \equiv 0\,, \quad \psi_2 = \frac{\sqrt{z\bar{z} + R^2}}{\sqrt[4]{(z-a)(\bar{z}-\bar{a}) + \varepsilon^2}\sqrt[4]{(z-b)(\bar{z}-\bar{b}) + \varepsilon^2}}\,. \tag{B.70}$$

This zero mode is normalizable with respect to the measure on $S^2$ which is given by $\Omega_R^2 \mathrm{d}x_1 \mathrm{d}x_2$.

# C Uniqueness of $\chi_2$ from periodicity properties

The existence of an analytic function with the periodicity properties (4.58) is seen by choosing

$$\chi_2(z) = \theta(z + \frac{i}{2}\tau - \frac{i}{2}\Im(z_1 + z_2), i\tau)\,, \tag{C.71}$$

cf. eq. (4.33). To show the uniqueness (up to a constant factor) of $\chi_2$ we assume that there is another analytic function $\tilde{\chi}_2$ satisfying eq. (4.58). Now consider the meromorphic function $f(z) := \tilde{\chi}_2(z)/\chi_2(z)$. This is an elliptic function with periods 1 and $i\tau$. But $f(z)$ has only a single pole in the fundamental domain ($0 < Re(z) < 1$ , $0 < \Im(z) < \tau$) at the zero of $\theta(z + \frac{i}{2}\tau - \frac{i}{2}\Im(z_1 + z_2), i\tau)$. But an elliptic function has to have at least two poles in the fundamental domain or it is a constant [22]. Hence, we infer that $f(z)$ is a constant and $\chi_2(z)$ is (up to a constant factor) given by (C.71).

# D Smeared out vortex configurations

Using the functions

$$\phi(z, z_k) = \frac{1}{4}\log\left(\theta^+(z, z_k)\overline{\theta^+(z, z_k)} + \theta^-(z, z_k)\overline{\theta^-(z, z_k)}\right), \tag{D.72}$$

$$\theta^\pm(z, z_k) = \theta(z + \frac{1}{2} + \frac{i}{2}\tau - z_k \pm \varepsilon, i\tau)\,, \quad \varepsilon \in \mathbb{R}_+\,, \tag{D.73}$$

the gauge potential $A$ of a smeared out center vortex centered at the point $z_k$ is given by

$$A_x = -i\partial_y(\phi(z, z_k))\,, \quad A_y = i\partial_x(\phi(z, z_k))\,, \tag{D.74}$$

from which we obtain

$$A_z = 1/2(A_x - iA_y) = \partial_z\phi(z, z_k) \tag{D.75}$$

$$= \frac{\overline{\theta^+(z, z_k)}\partial_z\theta^+(z, z_k) + \overline{\theta^-(z, z_k)}\partial_z\theta^-(z, z_k)}{\theta^+(z, z_k)\overline{\theta^+(z, z_k)} + \theta^-(z, z_k)\overline{\theta^-(z, z_k)}}\,. \tag{D.76}$$

By Taylor expanding around $z_k$ we obtain the behavior of the gauge potential for small distances $r$ from the center $z_k$ of the vortex:

$$A_x = -y\frac{i}{2(r^2 + \varepsilon^2)}\,, \quad A_y = x\frac{i}{2(r^2 + \varepsilon^2)}\,. \tag{D.77}$$

This gauge potential indeed represents a smeared out center vortex. In the limit $\varepsilon \to 0$ $A$ becomes the gauge potential of an ideal center vortex [15] on the torus.

The gauge potential of two smeared out center vortices at the points $z_{1/2}$ is simply given by the sum of two one-vortex gauge potentials (D.75), i.e.

$$\begin{aligned} A_z &= \partial_z(\phi(z,z_1)+\phi(z,z_2)) \\ &= \sum_{k=1}^{2} \frac{\overline{\theta^+(z,z_k)}\partial_z\theta^+(z,z_k)+\overline{\theta^-(z,z_k)}\partial_z\theta^-(z,z_k)}{\theta^+(z,z_k)\overline{\theta^+(z,z_k)}+\theta^-(z,z_k)\overline{\theta^-(z,z_k)}} \,. \end{aligned} \tag{D.78}$$

It turns out that in this vortex background there is only one normalizable fermionic zero mode given by

$$\psi_1 \equiv 0\,, \quad \psi_2 = \left(\prod_{k=1}^{2}\left(\theta^+(z,z_k)\overline{\theta^+(z,z_k)}+\theta^-(z,z_k)\overline{\theta^-(z,z_k)}\right)\right)^{-1/4}\overline{\chi_2(z)}\,, \tag{D.79}$$

where $\chi_2(z)$ has to be an analytic function with periodicity properties given by eq. (4.58). These periodicity properties fix $\chi_2(z)$ up to a factor to be a theta function:

$$\chi_2(z) = \theta(z+\frac{i}{2}\tau-\frac{i}{2}\Im(z_1+z_2), i\tau)\,. \tag{D.80}$$

The zeros of this function are at the points $z = \frac{1}{2}+\frac{i}{2}\Im(z_1+z_2)+m+ni\tau$ , $m,n\in\mathbb{Z}$, i.e. independent of the real parts of $z_1$ and $z_2$. The probability density of the zero mode is again peaked at the locations of the two vortices, i.e. at $z_1$ and $z_2$, cf. fig. 3.

# E Multi vortex solution on $\mathbb{T}^2$

Let us assume we have a number of thick vortices and anti-vortices at the points $z_k$ , $k = 1,\ldots,m^+$ and $z_l$ , $l = m^++1,\ldots,m^++m^-$, respectively, where the total number $m^+ + m^- = m$ of vortices is even. The corresponding gauge potential reads

$$A_z = \frac{1}{4}\partial_z\left(\sum_{k=1}^{m^+}\phi(z,z_k) - \sum_{l=m^++1}^{m^++m^-}\phi(z,z_l)\right)\,, \tag{E.81}$$

where $\phi(z,z_k)$ is defined by eq. (D.72). The fermionic zero modes $\psi_{1/2}(z)$ are given by

$$\begin{aligned} \psi_1 &= \prod_{k=1}^{m^+}\left(\theta^+(z,z_k)\overline{\theta^+(z,z_k)}+\theta^-(z,z_k)\overline{\theta^-(z,z_k)})\right)^{1/4}\times \\ &\quad\times\prod_{l=m^++1}^{m^++m^-}\left(\theta^+(z,z_l)\overline{\theta^+(z,z_l)}+\theta^-(z,z_l)\overline{\theta^-(z,z_l)})\right)^{-1/4}\chi_1(z)\,, \\ \psi_2 &= \prod_{k=1}^{m^+}\left(\theta^+(z,z_k)\overline{\theta^+(z,z_k)}+\theta^-(z,z_k)\overline{\theta^-(z,z_k)})\right)^{-1/4}\times \end{aligned}$$

$$\times \prod_{l=m^++1}^{m^++m^-} \left(\theta^+(z,z_l)\overline{\theta^+(z,z_l)} + \theta^-(z,z_l)\overline{\theta^-(z,z_l)})\right)^{1/4} \overline{\chi_2(z)}\,,$$

where $\theta^\pm$ is defined by eq. (D.73). The periodicity properties of $\psi_{1/2}$, cf. eq. (4.50), define the periodicity properties of the analytic functions $\chi_{1/2}(z)$:

$$\begin{aligned}
\chi_1(z+i\tau) &= e^{-\left(((\Delta m/2)(\pi\tau - 2\pi i(z+i\tau/2)) - \pi\Im\left(\sum_{k=1}^{m^+} z_k - \sum_{l=m^++1}^{m} z_l\right)\right)}\chi_1(z)\,, && \text{(E.82)}\\
\chi_1(z+1) &= \chi_1(z)\,, && \text{(E.83)}\\
\chi_2(z+i\tau) &= e^{\left(((\Delta m/2)(\pi\tau - 2\pi i(z+i\tau/2)) - \pi\Im\left(\sum_{k=1}^{m^+} z_k - \sum_{l=m^++1}^{m} z_l\right)\right)}\chi_2(z)\,, && \text{(E.84)}\\
\chi_2(z+1) &= \chi_2(z)\,, && \text{(E.85)}
\end{aligned}$$

where $\Delta m = m^+ - m^-$ and $m = m^+ + m^-$. If $\Delta m > 0$ we find a $\Delta m/2$-dimensional vector space of left-handed zero modes $\psi_2$ with analytic functions $\chi_2(z)$. In the case $\Delta m < 0$ we find a $\Delta m/2$-dimensional vector space of right-handed zero modes with analytic functions $\chi_1(z)$. The functions $\chi_{1/2}$ are then given by products of theta functions. In the case $\Delta m > 0$ we obtain

$$\chi_2(z) = \prod_{j=1}^{\Delta m/2} \theta(z + i\tau/2 - \tilde{z}_j, i\tau)\,, \tag{E.86}$$

where the complex numbers $\tilde{z}_j$ have to fulfill the conditions

$$\Re(\sum_{j=1}^{\Delta m/2} \tilde{z}_j) = 0\,, \quad \Im(\sum_{j=1}^{\Delta m/2} \tilde{z}_j) = \frac{1}{2}\Im\left(\sum_{k=1}^{m^+} z_k - \sum_{l=m^++1}^{m} z_l\right). \tag{E.87}$$

The set of functions $\chi_2$ given by eqs. (E.86, E.87) forms an $\Delta m/2$-dimensional vector space.

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